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SELF-GRAVITATING STRING-LIKE CONFIGURATIONS FROM NONLINEAR ELECTRODYNAMICS

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We consider static, cylindrically symmetric configurations in general relativity coupled to nonlinear electrodynamics (NED) with an arbitrary gauge-invariant Lagrangian of the form $L_{\rm em} = \Phi(F)$, $F = F_{\alpha\beta}F^{\alpha\beta}$. We study electric and magnetic fields with three possible orientations: radial (R), longitudinal (L) and azimuthal (A), and try to find solitonic stringlike solutions, having a regular axis and a flat metric at large r, with a possible angular defect. Assuming that the function $\Phi(F)$ is regular at small F, it is shown that a regular axis is impossible in R-fields if there is a nonzero effective electric charge and in A-fields if there is a nonzero effective electric current along the axis. Thus solitonic solutions are only possible for purely magnetic R-fields and purely electric A-fields, in cases when $\Phi(F)$ tends to a finite limit at large F. For both R- and A-fields it is shown that the desired large r asymptotic is only possible with a non-Maxwell behaviour of $\Phi(F)$ at small F. For L-fields, solutions with a regular axis are easily obtained (and can be found by quadratures) whereas a desired large r asymptotic is only possible in an exceptional solution; the latter gives rise to solitonic configurations in case $\Phi(F) = \mathrm{const}\cdot\sqrt{F}$. We give an explicit example of such a solution.

1. Introduction

Self-gravitating systems with classical nonlinear field sources find many applications in modern unification theory and cosmology. On the one hand, there naturally appear various scalar fields with nonlinear potentials and different coupling to gravity, on the other, some models of string theory create in their low-energy limits such theories as the Born-Infeld nonlinear electrodynamics or its non-Abelian modifications [1]. Of utmost importance in any such theory are solitonic or solitonlike solutions, combining global regularity and localization in a certain number of dimensions, such as particlelike solutions, kinks, domain walls, cosmic strings, etc.

Of particular interest are cylindrically symmetric (stringlike) solutions, with fields localized in a vicinity of the symmetry axis. Nielsen and Olesen [2] pointed out that vortex solutions of the Higgs model behave classically as Nambu dual strings [3] in the strong-coupling limit. As an example, they have shown that a Higgs-type Lagrangian allows for vortex-lines solutions, similar to vortex lines in a type II superconductor. Self-gravitating cosmic strings have been a subject of great interest in cosmology in the recent decades, see Refs. [5, 6] for reviews.

Stringlike configurations of nonlinear fields have also been studied: in Ref. [7] for self-interacting scalar fields and in Ref. [8] for interacting scalar and electromagnetic fields.

The present paper continues this trend and studies the possibility of obtaining regular self-gravitating stringlike configurations of a nonlinear electromagnetic field with an arbitrary gauge-invariant Lagrangian $\Phi(F)$, $F = F_{\mu\nu}F^{\mu\nu}$. We consider three possible types of electromagnetic fields compatible with cylindrical symmetry: radial (R), azimuthal (A) and longitudinal (L) fields. We formulate the conditions of the solutions' regularity on the symmetry axis and for their proper behaviour far from the axis and conclude that solitonic stringlike configurations, i.e., those well-behaved both on the axis and at infinity, can be built in the cases of R- and A-fields under certain conditions upon the function $\Phi(F)$. The regular axis conditions are somewhat similar to the regular centre conditions for the NED-Einstein system in the case of spherical symmetry [9].

According to [9], it is possible to obtain regular asymptotically flat purely magnetic solutions (black holes and monopoles) in NED which reduces to the Maxwell theory ($\Phi \sim F$) at small F. Unlike that, regular (flat or string) asymptotics of cylindrically symmetric configurations are only possible when $\Phi(F) = o(F)$ at small F. This circumstance is closely related to the fact that the corresponding Einstein-Maxwell solutions are not well-behaved at infinity.

For L-fields, although a regular axis is easily obtained (even in the Einstein-Maxwell system [10, 11]), the general solution with a regular axis (which is obtained by quadratures) cannot lead to a stringlike solitonic configuration since a regular spatial asymptotic is absent. There is, however, an exceptional purely magnetic solution with $\Phi(F) = \mathrm{const}\cdot\sqrt{F}$ which turns out to be solitonic under some further restrictions.

It should be stressed that, in nonsingular configu-

rations of any spatial symmetry, effective electric and magnetic charges, characterizing the field behaviour at large (e.g., at a spatial asymptotic), appear in nonlinear theory without postulating the existence of such charges in the initial formulation of the theory (as is done in the Maxwell theory where the current density j^μ is introduced in the interaction term $j^\mu A_\mu$, added to the field Lagrangian $\sim F^{\mu\nu}F_{\mu\nu}$). It is the field nonlinearity that leads to effective electric and magnetic charge densities or currents distributed in space. In this way magnetic monopoles and regular black holes with magnetic charges were obtained in Ref. [9] without introducing a magnetic charge density as a separate quantity; their cylindrically symmetric analogues are solitonic configurations whose existence is considered here.

2. Field equations and regularity conditions

2.1. Field equations

Consider gauge-invariant NED in general relativity (GR), with the action

$$S = \frac{1}{2æ}\int \sqrt{-g}\, d^4x[R - G\Phi(F)], \tag{1}$$

where R is the scalar curvature, g is the determinant of the metric tensor $g_{\mu\nu}$, G is the gravitational constant, $æ = 8\pi G$, and $\Phi(F)$ is an arbitrary function of the invariant $F = F_{\alpha\beta}F^{\alpha\beta}$. Maxwell's electrodynamics corresponds to $\Phi(F) \equiv F$, and, in nonlinear theory, it is natural to assume a Maxwell behaviour ($\Phi(F) \approx F$) at small F.

The electromagnetic field equations following from (1) and the Bianchi identities for $F_{\mu\nu}$ have the form

$$\nabla_\nu(F^{\mu\nu}\Phi_F) \ = \ 0, \qquad \Phi_F \equiv \frac{d\Phi}{dF}, \tag{2}$$

$$\nabla_\nu {}^*F^{\mu\nu} \ = \ 0, \quad {}^*F^{\mu\nu} = -\frac{1}{2\sqrt{-g}}\varepsilon^{\lambda\rho\mu\nu}F_{\lambda\rho}, \tag{3}$$

where $\varepsilon^{\lambda\rho\mu\nu}$ is the Levi-Civita symbol and $*$ denotes the Hodge dual.

Variation of (1) with respect to $g^{\mu\nu}$ leads to the Einstein equations

$$G^\nu_\mu \equiv R^\nu_\mu - \tfrac{1}{2}\delta^\nu_\mu R = -æT^\nu_\mu \tag{4}$$

with the electromagnetic stress-energy tensor (SET)

$$T^\nu_\mu = \frac{1}{16\pi}[-4F_{\mu\alpha}F^{\nu\alpha}\Phi_F + \delta^\nu_\mu\Phi]. \tag{5}$$

A static, cylindrically symmetric metric can be written as

$$ds^2 = e^{2\gamma}dt^2 - e^{2\alpha}du^2 - e^{2\xi}dz^2 - e^{2\beta}d\varphi^2, \tag{6}$$

where α, β, γ, ξ are functions of the radial coordinate u only; $z \in \mathbb{R}$ and $\varphi \in [0; 2\pi)$ are the longitudinal and azimuthal coordinates, respectively. There is still freedom of choosing the u coordinate.

We find it convenient to use the coordinate condition [11]

$$\alpha = \gamma + \xi + \beta, \tag{7}$$

so that u is a harmonic coordinate. Its range is not specified until the full geometry is known. Under the condition (7), Eqs. (4) take the following symmetric form:

$$\begin{aligned} \beta'' + \xi'' - U \ &= \ -æT^0_0\, e^{2\alpha}, && (8)\\ U \ &= \ -æT^1_1\, e^{2\alpha}, && (9)\\ \beta'' + \gamma'' - U \ &= \ -æT^2_2\, e^{2\alpha}, && (10)\\ \gamma'' + \xi'' - U \ &= \ -æT^3_3\, e^{2\alpha}, && (11)\\ U \ &\equiv \ \beta'\gamma' + \beta'\xi' + \gamma'\xi'. \end{aligned}$$

We will seek solutions in three cases compatible with cylindrical symmetry, with the following nonzero components of $F_{\mu\nu}$ in each case:

Radial (R) fields: electric, $F_{01}(u)$ ($E^2 = F_{01}F^{10}$), and magnetic, $F_{23}(u)$ ($B^2 = F_{23}F^{23}$).

Azimuthal (A) fields: electric, $F_{03}(u)$ ($E^2 = F_{03}F^{30}$), and magnetic, $F_{12}(u)$ ($B^2 = F_{12}F^{12}$).

Longitudinal (L) fields: electric, $F_{02}(u)$ ($E^2 = F_{02}F^{20}$), and magnetic, $F_{13}(u)$ ($B^2 = F_{13}F^{13}$).

Here E and B are the absolute values of the electric field strength and magnetic induction, respectively.

2.2. Regularity on the axis and regular (flat or string) spatial asymptotics

From the whole set of solutions to the field equations, we will try to single out those having (i) a regular axis of symmetry and (ii) such a behaviour far from the axis that corresponds to the gravitational field of an isolated cylindrically symmetric matter distribution or a cosmic string, i.e., a flat or string asymptotic at spatial infinity (in what follows, a *regular* asymptotic, for short).

Let us first write the regularity conditions without specifying the u coordinate.

The regularity conditions on the axis, i.e. for $u = u_a$ such that $r \equiv e^\beta \to 0$, include the finiteness requirement for the algebraic curvature invariants and the condition

$$|\beta'|\, e^{\beta-\alpha} \to 1 \tag{12}$$

(the prime denotes d/du), expressing the absence of a conical singularity.

Among the curvature invariants it is sufficient to deal with the Kretschmann scalar $K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ which, for the metric (6), is a sum of all squared nonzero Riemann tensor components $R^{\mu\nu}{}_{\rho\sigma}$:

$$K \ = \ 4\sum_{i=1}^{6} K_i^2;$$

$$
\begin{aligned}
K_1 &= R^{01}{}_{01} = -\,\mathrm{e}^{-\alpha-\gamma}(\gamma'\,\mathrm{e}^{\gamma-\alpha})',\\
K_2 &= R^{02}{}_{02} = -\,\mathrm{e}^{-2\alpha}\gamma'\xi',\\
K_3 &= R^{03}{}_{03} = -\,\mathrm{e}^{-2\alpha}\beta'\gamma',\\
K_4 &= R^{12}{}_{12} = -\,\mathrm{e}^{-\alpha-\xi}(\xi'\,\mathrm{e}^{\xi-\alpha})',\\
K_5 &= R^{13}{}_{13} = -\,\mathrm{e}^{-\alpha-\beta}(\beta'\,\mathrm{e}^{\beta-\alpha})',\\
K_6 &= R^{23}{}_{23} = -\,\mathrm{e}^{-2\alpha}\beta'\xi'
\end{aligned}
\qquad (13)
$$

For $K < \infty$ it is thus necessary and sufficient that all $|K_i| < \infty$, and this in turn guarantees that all algebraic invariants of the Riemann tensor will be finite. Note that all K_i, as well as the condition (12), are invariant under reparametrization of u.

It can be verified (see [7] for details) that the regular axis conditions hold at $u = u_{\rm ax}$, where $r = \mathrm{e}^{\beta} \to 0$, if and only if

$$
\gamma = \gamma_{\rm ax} + O(r^2); \qquad \xi = \xi_{\rm ax} + O(r^2), \qquad (14)
$$

$$
|\beta'|\,\mathrm{e}^{\beta-\alpha} = 1 + O(r^2) \qquad (15)
$$

as $u \to u_{\rm ax}$. Here and henceforth the symbol $O(f)$ denotes a quantity either of the same order of magnitude as f in a certain limit, or smaller, while the symbol $\sim$ connects quantities of the same order of magnitude.

A useful necessary condition for regularity follows from the Einstein equations. At points of a regular axis, as at any regular point, the curvature invariants R and $R_{\mu\nu}R^{\mu\nu}$ are finite. Since the Ricci tensor for the metric (6) is diagonal, the invariant $R_{\mu\nu}R^{\mu\nu} \equiv R_\mu^\nu R_\nu^\mu$ is a sum of squares, hence each component R_μ^μ (no summing) is finite. Then, due to the Einstein equations, each component of the SET T_μ^ν is finite as well:

$$
|T_\mu^\nu| < \infty. \qquad (16)
$$

Thus, requiring only regularity of the geometry, we obtain, as its necessary condition, the finiteness of all SET components with mixed indices. This is true not only for the present case, but always when R_μ^ν is diagonal.

Let us now formulate the conditions at regular spatial asymptotics. We require the existence of a spatial infinity, i.e., $u = u_\infty$ such that $r = \mathrm{e}^{\beta} \to \infty$, where the metric is either flat, or corresponds to the gravitational field of a cosmic string.

Then, first, as $u \to u_\infty$, a correct behaviour of clocks and rulers requires $|\gamma| < \infty$ and $|\xi| < \infty$ or, choosing proper scales along the t and z axes, one can write

$$
\gamma \to 0, \quad \xi \to 0 \quad \text{as} \quad u \to u_\infty. \qquad (17)
$$

Second, at large r the condition (12) should be replaced with a more general one,

$$
\begin{aligned}
&|\beta'|\,\mathrm{e}^{\beta-\alpha} \to 1-\mu,\\
&\mu = \text{const} < 1, \quad \text{as} \quad u \to u_\infty,
\end{aligned}
\qquad (18)
$$

so that the circumference-to-radius ratio for the circles $u = \text{const}$, $z = \text{const}$ tends to $2\pi(1-\mu)$ instead of 2π, μ being the angular defect. Under the asymptotic conditions (17), (18), $\mu > 0$, the solution can simulate a cosmic string. A flat asymptotic takes place if $\mu = 0$.

Third, the curvature tensor should vanish at large r, and, due to the Einstein equations, all SET components must decay quickly enough. It can be easily checked that the conditions (17) and (18) automatically imply that all $K_i = o(r^{-2})$ where K_i are defined in (13). Consequently the same decay rate at a regular asymptotic takes place in all components of T_μ^ν, and one can verify, in particular, that the total material field energy per unit length along the z axis is finite:

$$
\int T_0^0 \sqrt{-{}^3g}\, d^3x = \int T_0^0\, \mathrm{e}^{\alpha+\beta+\xi}\, du\, dz\, d\phi < \infty \qquad (19)
$$

where integration in z covers a unit interval. A similar condition in flat-space field theory is used as a criterion of field energy being localized around the symmetry axis, which is one of the requirements to solitonic solutions. The set of asymptotic regularity requirements (17), (18) for self-gravitating solutions is thus much stronger than (19) and contains the latter as a corollary.

Specifically, if we use the harmonic u coordinate, it is easily shown that both a regular axis and a regular spatial asymptotic require $u \to \pm\infty$. Choosing, without loss of generality, $u_{\rm ax} = -\infty$ and $u_\infty = +\infty$, one finds that at a regular axis, in addition to (14) and (15),

$$
r = \mathrm{e}^{\beta} \sim \mathrm{e}^{cu}, \quad c = \mathrm{e}^{\gamma+\xi}\Big|_{u\to-\infty} = \text{const} > 0. \qquad (20)
$$

At a regular spatial asymptotic ($u \to +\infty$), in addition to (17) and (18), we have

$$
\mathrm{e}^{\alpha} \approx \mathrm{e}^{\beta} = r \sim \mathrm{e}^{(1-\mu)u}, \qquad (21)
$$

and the SET components T_μ^ν must decay at large u quicker than $r^{-2} \sim \mathrm{e}^{-2(1-\mu)u}$.

3. Radial electromagnetic fields

The Einstein-Maxwell equations for a radial electromagnetic field lead to the metric [11]

$$
\begin{aligned}
ds^2 = \frac{K dt^2}{s^2(h,u)} - \frac{s^2(h,u)}{K}\big[\,\mathrm{e}^{2(a+b)u}du^2\\
+\,\mathrm{e}^{2au}dz^2\big] + \mathrm{e}^{2bu}d\varphi^2,
\end{aligned}
\qquad (22)
$$

where $K = (Gq^2)^{-1}$, $q^2 = q_e^2 + q_m^2$, $a,\ b = \text{const}$, the function $s(h,u)$ is defined as

$$
s(h,u) = \begin{cases} h^{-1}\sinh(hu), & h > 0,\\ u, & h = 0,\\ h^{-1}\sin(hu), & h < 0, \end{cases} \qquad (23)
$$

and $h^2\,\mathrm{sign}\,h = ab$. The electromagnetic field is represented by

$$
F^{01} = q_e\,\mathrm{e}^{-2\alpha}, \qquad F_{23} = q_m, \qquad (24)
$$

the constants q_e and q_m being the linear electric and magnetic charges, respectively.

This space-time possesses a singular axis (charged thread) and spatial infinity if $h \geq 0$ and $h+B > 0$. The large r asymptotic then cannot be regular, in particular, $g_{tt} \to 0$, and $|\beta'|\,\mathrm{e}^{\beta-\alpha} \sim \mathrm{e}^{-au}$ does not tend to a finite constant. In other cases we have two singular axes.

Our more general cylindrically symmetric system with nonlinear radial electric and magnetic fields closely resembles a system with the same Lagrangian (1) in the spherically symmetric case [9]. It has been shown in Ref. [9] that the field system (1), with any function $\Phi(F)$ having a Maxwell asymptotic at small F, does not admit a static, spherically symmetric solution with a regular centre and a nonzero electric charge. We shall see that a similar theorem can be proved using the same arguments for radial fields and the metric (6). Moreover, *both here and in the spherically symmetric case*, instead of requiring a Maxwell asymptotic, *it is sufficient to assume that the function $\Phi(F)$ is simply smooth at $F = 0$.*

Indeed, Eqs. (2) and (3) give

$$F^{01}\Phi_F = q_e\,\mathrm{e}^{-2\alpha}, \qquad F_{23} = q_m, \tag{25}$$

where, as before, the constants q_e and q_m are the linear electric and magnetic charges. It follows that

$$E^2\Phi_F^2 = q_e^2\,\mathrm{e}^{-2\xi-2\beta}, \tag{26}$$

$$B^2 = q_m^2\,\mathrm{e}^{-2\xi-2\beta}, \tag{27}$$

and the nonzero SET components are

$$T_0^0 = T_1^1 = \frac{1}{16\pi}(4E^2\Phi_F + \Phi), \tag{28}$$

$$T_2^2 = T_3^3 = \frac{1}{16\pi}(-4B^2\Phi_F + \Phi). \tag{29}$$

Theorem 1. *The system (1), such that $\Phi(F)$ is C^1-smooth at $F = 0$, does not admit a static, cylindrically symmetric solution for radial electromagnetic fields with a regular axis and a nonzero electric charge.*

Proof. From the condition $T_\mu^\nu < \infty$ it follows

$$(E^2 + B^2)|\Phi_F| < \infty. \tag{30}$$

Suppose first that $q_m = 0$, $q_e \neq 0$. Therefore, from (26) and (30) it follows that, at a regular axis, $E^2\Phi_F$ is finite whereas $E^2\Phi_F^2 \to \infty$. These conditions combined lead to $F \to 0$ and $\Phi_F \to \infty$, that is, a singular behaviour of Φ at small F. Thus for purely electric fields the theorem is valid.

Suppose now $q_m \neq 0$ and $q_e \neq 0$. Then (30) should hold for E^2 and B^2 taken separately. As stated previously, this condition applied to E^2, combined with (26), leads to $\Phi_F \to \infty$. But B^2 also tends to infinity, so even stronger $B^2\Phi_F \to \infty$, violating (30). The theorem is proved.

Consider the remaining case of a purely magnetic radial field. Then, according to (28),

$$T_0^0 = T_1^1 = \Phi/(16\pi). \tag{31}$$

Meanwhile, by Eq. (27), $F = 2B^2 \to \infty$ at a possible regular axis where $\xi \to \mathrm{const}$ and $\beta \to -\infty$, therefore the necessary condition (16) for regularity on the axis holds as long as $\Phi \to \mathrm{const}$ as $F \to +\infty$. Moreover, it then follows that $\Phi_F = o(1/F)$, hence $B^2\Phi_F \to 0$, which means that the SET (28), (29) has the structure of a cosmological term: $T_\mu^\nu \sim \Phi\delta_\mu^\nu$ near a regular axis. We conclude that *a possible solution with a radial magnetic field, possessing a regular axis, is approximately de Sitter or anti-de Sitter near such an axis.*

As for a regular spatial asymptotic, we have seen that it is absent in the Einstein-Maxwell solution (except for the trivial case of flat space and zero $F_{\mu\nu}$). The same will evidently be true if $\Phi(F)$ has a Maxwell behaviour, $\Phi(F) \sim F$, at small F: the NED-Einstein solution will then behave at large r as an Einstein-Maxwell solution.

As follows from (28) and (29), a regular spatial asymptotic requires $F\Phi_F = o(1/r^2)$ at large r. For a purely magnetic solution, in case $\xi \to \mathrm{const}$, we have $F \sim 1/r^2$, hence $\Phi_F = o(1)$. This is a necessary condition of regularity. Another evident necessary condition is $\Phi = o(1/r^2)$.

Let us return to the Einstein equations. Due to $T_2^2 = T_3^3$, they lead to

$$\beta'' - \xi'' = 0 \;\Rightarrow\; \beta(u) = \xi(u) + c_1 u, \tag{32}$$

where $c_1 = \mathrm{const}$ and another integration constant is set equal to zero by choosing the scale along the z axis. To have a regular axis at $u \to -\infty$, we should put $c_1 > 0$.

A sum of (8) and (9) leads to

$$2\beta'' = -G(4E^2\Phi_F + \Phi)\,\mathrm{e}^{2\alpha}. \tag{33}$$

We can also obtain a relation between $\beta(u)$ and $\gamma(u)$: the difference of (8) and (9) gives

$$\beta'' - \beta'^2 + \beta' c_1 + \gamma' c_1 - 2\beta'\gamma' = 0. \tag{34}$$

It is hard to obtain an exact solution to the NED-Einstein equations with a given function $\Phi(F)$. However, for a purely magnetic field, the general solution can be described by specifying $\beta(u)$ if we consider $\Phi(F)$ as one of the unknown functions. Indeed, one then finds $\xi(u)$ and $\gamma(u)$ from (32) and (34), then $\alpha(u)$ from (7), $\Phi(u)$ from (33) and $F(u)$ from (27). A comparison of the latter two functions leads to $\Phi(F)$.

We conclude that solitonic solutions, regular both on the axis and at infinity, are not excluded with radial magnetic fields. Necessary conditions for obtaining such a solution are: (i) $q_e = 0$, $q_m \neq 0$; (ii) $\Phi(F) \to \mathrm{const}$ as $F \to +\infty$ and (iii) $\Phi = o(F)$ as $F \to 0$ (a non-Maxwell behaviour at small F).

One more necessary condition follows from (32) and (33). Namely, due to (32), $\beta' = c_1$ both on the axis and on the asymptotic, and integration of (33) gives:

$$\int_{-\infty}^{+\infty} \Phi\,\mathrm{e}^{2\alpha} du = 0, \tag{35}$$

whence it follows that $\Phi(F)$ should have an alternating sign in the range $F = 2B^2 > 0$ corresponding to $u \in \mathbb{R}$.

It should be stressed that all these conditions are only necessary. Even if all of them hold, one cannot guarantee that a given function $\beta(u)$ will lead to a valid solitonic solution. Thus, a nontrivial requirement is that the resulting $\Phi(F)$ should be a function. In particular, if the function $F = 2B^2(u)$ obtained from (27) is monotonic, then $\Phi(F)$ will be a function only if $\Phi(u)$ obtained from (33) (with $E = 0$) is also monotonic. This in turn requires (according to the Einstein equations) that the difference $\beta'' - \gamma''$ should have an invariable sign at all u.

4. Azimuthal electromagnetic fields

The NED-Einstein equations for azimuthal electromagnetic fields can be studied in quite a similar manner as for radial fields. We therefore only mention the main points, avoiding the details.

The Einstein-Maxwell equations for an azimuthal electromagnetic field lead to the metric [11]

$$ds^2 = \frac{\cosh^2(hu)}{Kh^2}\big[\,\mathrm{e}^{2bu}dt^2 - \mathrm{e}^{2(a+b)u}du^2 - \mathrm{e}^{2au}d\varphi^2\big] - \frac{Kh^2}{\cosh^2(hu)}dz^2, \qquad (36)$$

where $K = [G(i_e^2 + i_m^2)]^{-1}$, $h^2 = ab$, $a, b = \text{const}$, $a > 0$, $b > 0$, and the electromagnetic field given by

$$F_{03} = i_m = \text{const}; \qquad F^{12} = i_e\,\mathrm{e}^{-2\alpha}, \quad i_e = \text{const}, \qquad (37)$$

where i_e and i_m are the effective currents of electric and magnetic charges along the z axis, respectively.

This solution, like (22), (24), does not provide a regular axis or a regular large r asymptotic for any values of the integration constants.

In NED under consideration, Eqs. (2) and (3) give

$$F_{03} = i_m, \qquad F^{12}\Phi_F = i_e\,\mathrm{e}^{-2\alpha}, \qquad (38)$$

which leads to

$$E^2 = i_m^2\,\mathrm{e}^{-2\gamma-2\beta}, \qquad (39)$$
$$B^2\Phi_F^2 = i_e^2\,\mathrm{e}^{-2\gamma-2\beta}. \qquad (40)$$

The SET components have the form

$$T_0^0 = T_3^3 = \frac{1}{16\pi}(4E^2\Phi_F + \Phi), \qquad (41)$$
$$T_1^1 = T_2^2 = \frac{1}{16\pi}(-4B^2\Phi_F + \Phi). \qquad (42)$$

Using the same arguments as in the case of radial fields, we can obtain an analogue of Theorem 1 for azimuthal fields:

Theorem 2. *The system (1), such that $\Phi(F)$ is C^1-smooth at $F = 0$, does not admit a static, cylindrically symmetric solution for azimuthal electromagnetic fields with a regular axis and a nonzero effective electric current i_e.*

Thus a regular axis is not excluded only in the case of an azimuthal electric field, which can result from an effective magnetic current.

All this has been obtained without using the Einstein equations. The latter give, in full similarity with Sec. 3:

$$\beta(u) = \gamma(u) + c_2 u, \quad c_2 = \text{const}, \qquad (43)$$

with $c_2 > 0$ if we wish to have a regular axis;

$$2\beta'' = G(4B^2\Phi_F - \Phi)\,\mathrm{e}^{2\alpha}, \qquad (44)$$

and

$$\beta'' - \beta'^2 + \beta' c_2 + \xi' c_2 - 2\beta'\xi' = 0. \qquad (45)$$

It is again hard to obtain an exact solution with a given function $\Phi(F)$. However, in case $i_e = 0$, the general solution may be parametrized by $\beta(u)$. Indeed, given $\beta(u)$, we can find $\gamma(u)$ and $\xi(u)$ from (43) and (45), then $\alpha(u)$ from (7), $\Phi(u)$ from (44) and $F(u)$ from (39). Comparing the latter two functions, we can find $\Phi(F)$.

A regular asymptotic again needs a non-Maxwell behaviour of $\Phi(F)$ at small F: $\Phi = o(F)$.

We conclude that solitonic solutions, regular both on the axis and at infinity, are not excluded with azimuthal electric fields. Necessary conditions for obtaining such a solution are: (i) $i_e = 0$, $i_m \neq 0$; (ii) $\Phi(F) \to \text{const}$ as $F \to -\infty$, (iii) $\Phi = o(F)$ as $F \to 0$, and (iv) an alternating sign of $\Phi(F)$ in the range $F = -2E^2 < 0$ corresponding to $u \in \mathbb{R}$. Again, as described above for R-fields, these conditions are not sufficient, and, in particular, a nontrivial problem is to provide that the resulting $\Phi(F)$ will be a function.

5. Longitudinal electromagnetic fields

The Einstein-Maxwell equations for a longitudinal electromagnetic field lead to the metric [11]

$$ds^2 = \frac{\cosh^2(hu)}{Kh^2}\big[\,\mathrm{e}^{2bu}dt^2 - \mathrm{e}^{2(a+b)u}du^2 - \mathrm{e}^{2au}dz^2\big] - \frac{Kh^2}{\cosh^2(hu)}d\varphi^2, \qquad (46)$$

where $K = [G(i_e^2 + i_m^2)]^{-1}$, $h^2 = ab$, $a, b = \text{const}$, $a > 0$, $b > 0$, and the electromagnetic field is given by

$$F_{02} = i_m = \text{const}; \qquad F^{13} = i_e\,\mathrm{e}^{-2\alpha}, \quad i_e = \text{const}, \qquad (47)$$

where i_e and i_m are the effective solinoidal electric and magnetic charges, respectively.

The metric (46) does not admit a spatial infinity since $g_{\varphi\varphi}$ is bounded above. Instead, there are two axes at $u \to \pm\infty$. The case $a = b = h$, $Kh^3 = 1$ corresponds to Melvin's nonsingular universe [10], where $u \to -\infty$ is

a regular axis while the other axis, $u = +\infty$, is infinitely remote.

In our NED-Einstein system, Eqs. (2) and (3) give

$$F_{02} = i_m, \qquad F^{13}\Phi_F = i_e\, \mathrm{e}^{-2\alpha}, \tag{48}$$

and

$$E^2 = i_m^2\, \mathrm{e}^{-2\gamma-2\xi}, \tag{49}$$
$$B^2\Phi_F^2 = i_e^2\, \mathrm{e}^{-2\gamma-2\xi}. \tag{50}$$

The SET components are

$$T_0^0 = T_2^2 = \frac{1}{16\pi}(4E^2\Phi_F + \Phi), \tag{51}$$
$$T_1^1 = T_3^3 = \frac{1}{16\pi}(-4B^2\Phi_F + \Phi). \tag{52}$$

The above Einstein-Maxwell solution shows that a regular axis in an L-field even exists with a linear electromagnetic field, therefore in NED it can also be easily obtained: if only the derivative Φ_F is finite at the corresponding value of F, a regular axis is obtained just as with the Maxwell field. Furthermore, assuming that there is a regular axis, the Einstein equations can be solved by quadratures for an arbitrary choice of $\Phi(F)$.

Indeed, the difference of Eqs. (8) and (10) leads to

$$\xi(u) = \gamma(u) + c_3 u, \qquad c_3 = \text{const}, \tag{53}$$

and to have a regular axis at $u \to -\infty$ we should put $c_3 = 0$. Then, the difference of Eqs. (9) and (10) gives

$$\xi'' - \xi'^2 - 2\beta'\xi' = 0, \tag{54}$$

whence

$$C\, \mathrm{e}^{2\beta} = \mathrm{e}^{-\xi}\xi', \qquad C = \text{const}. \tag{55}$$

Then, provided $C \neq 0$, Eq. (9) with (52) leads to

$$C\xi'' = -æT_1^1\, \mathrm{e}^{3\xi}\xi'. \tag{56}$$

Now, due to $\xi = \gamma$, the SET components can be obtained as functions of ξ: since

$$F = 2(B^2 - E^2) = 2\, \mathrm{e}^{-4\xi}[i_e^2\Phi_F^{-2} - i_m^2],$$

ξ is expressed in terms of F when $\Phi(F)$ is specified. Then F and $\Phi(F)$, and hence T_1^1, are determined as functions of ξ. As a result, Eq. (56) can be solved by quadratures:

$$u = -\int \frac{Cd\xi}{æ}\left(\int T_1^1\, \mathrm{e}^{3\xi} d\xi\right)^{-1}. \tag{57}$$

Regularity on the axis $u = -\infty$ can be achieved by adjusting the integration constants as long as $\Phi(F)$ is a smooth function since, with finite $\xi = \gamma$ and Φ_F, the strengths E and B are finite according to (49) and (50).

It turns out, however, that the above solution with $\xi = \gamma$ cannot possess a regular spatial asymptotic. Indeed, as follows from (55), $(\mathrm{e}^{-\xi})' \to \infty$ as $\beta \to \infty$, whereas the requirement (17) leads to $(\mathrm{e}^{-\xi})' \to 0$. In particular, solutions (53), (55) with $C \neq 0$, having a regular axis, cannot have a regular asymptotic.

It remains to consider the special case $C = 0$ in (55), when we can put $\xi \equiv \gamma \equiv 0$ without loss of generality. Then Eqs. (49) and (50), leading to

$$E^2 = i_m^2, \qquad B^2\Phi_F^2 = i_e^2, \tag{58}$$

make it possible to determine the function $\Phi(F)$:

$$\Phi^2(F) = 8i_e^2(F + 2i_m^2), \tag{59}$$

with an evidently non-Maxwell behaviour at small F. Now, the Einstein equations (9) and (11) lead to $T_1^1 = T_3^3 \equiv 0$, and one can easily verify that this condition is satisfied by $\Phi(F)$ taken according to Eq. (59). The only remaining field equation to be solved is

$$\mathrm{e}^{-2\beta}\beta'' = -\frac{4Gi_e^2}{\Phi}(F + 4i_m^2), \tag{60}$$

which follows from (8) or (10). Eq. (60) connects two unknown functions $F(u)$ and $\beta(u)$, so that one function may be chosen arbitrarily.

This choice makes it possible to obtain solitonic solutions, but only with a purely magnetic field created by the effective current i_e (one can check that a solution with $i_m \neq 0$ cannot have a regular asymptotic). We now have from (60)

$$G\Phi = \pm G\sqrt{8i_e^2F} = -2\beta''\, \mathrm{e}^{-2\beta}. \tag{61}$$

Fixing the sign on the l.h.s., we obtain that β'' has the same sign at all u, hence, in a nontrivial solution, $\beta'(+\infty) \neq \beta'(-\infty)$. With $\gamma \equiv \xi \equiv 0$, a regular axis corresponds to $\beta'(-\infty) = 1$, and, at a regular asymptotic, $\beta'(+\infty) = 1 - \mu$. Therefore, (i) a nontrivial solution inevitably leads to $\mu \neq 0$ and (ii) the angular defect $\mu > 0$ is obtained by choosing $+$ in Eq. (61).

Let us give a particular example of a solitonic solution belonging to this exceptional family. We choose

$$\mathrm{e}^{2\beta} = r_0^2\, \mathrm{e}^{(2-\mu)u}(\cosh ku)^{-\mu/k}, \tag{62}$$

with the constants r_0 (an arbitrary length), $\mu < 1$ (the angular defect) and $k \geq 1$ (which is required by the regularity condition (15)). Then Eq. (61) gives

$$G\sqrt{8i_e^2F} = \frac{k\mu}{2r_0^2}\, \mathrm{e}^{(\mu-2)u}(\cosh ku)^{-2-\mu/k}. \tag{63}$$

As is easily verified, F turns to zero as $u \to \pm\infty$, and the sufficient conditions of regularity are satisfied.

6. Concluding remarks

We have considered static cylindrically symmetric NED-Einstein equations with an arbitrary gauge-invariant NED Lagrangian of the form $\Phi(F)$.

For R-fields we have proved a theorem that a regular axis is impossible if the electric charge is nonzero and the function $\Phi(F)$ is regular at $F = 0$. A similar theorem

for A-fields tells us that a regular axis is impossible if there is a nonzero effective longitudinal electric current and $\Phi(F)$ is regular at $F = 0$. For both R- and A-fields, regular stringlike solutions are not excluded; they should possess a nonzero effective magnetic charge or a nonzero effective longitudinal magnetic current, respectively. Necessary conditions for the existence of such solutions are, among others, (i) a non-Maxwell behaviour of $\Phi(F)$ at small F and (ii) an alternating sign of $\Phi(F)$.

For L-fields, there exist configurations having a regular axis, and a general exact solution for such configurations was found by quadratures: specifying $\Phi(F)$, we can obtain all metric functions. In other cases, we have found parametrizations of the general solution in terms of one of the metric functions: knowing it, we can find other metric functions and $\Phi(F)$. A certain problem is then to obtain $\Phi(F)$ as a (single-valued) function since Φ and F are found separately as functions of the radial coordinate. It is for this reason that we do not give any explicit examples of solitonic solutions for R- and A-fields.

We have found that for L-fields, in addition to the above general solution, there is an exceptional solution ($C = 0$ in Eq. (55)) for a special choice (59) of $\Phi(F)$. Its properties are drastically different from those of the general solution, and, in particular, it gives rise to purely magnetic solitonic configurations with a nonzero angular defect μ. A specific example of such a configuration is given by Eqs. (62), (63).

In the present study, just as in the case of spherical symmetry [9], we find that the properties of electric and magnetic fields are quite different. In particular, only purely magnetic R-fields and purely electric A-fields can form regular self-gravitating configurations. This is a clear manifestation of the absence, in nonlinear theory, of the electric-magnetic duality which is so important in the Maxwell theory. It can be of interest to study another symmetry, the so-called FP duality [9] between different theories of NED, for various cylindrically symmetric configurations; this, however, goes beyond the scope of this paper.

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